Application of the Caputo Fractional Domain in Stage Structured Predator-Prey Dynamics

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Abstract: One of the most exciting fields of applied mathematics in recent years has been the study of biological species behavior. The significance of protecting endangered species in nature has led academics from a variety of disciplines to focus on this problem. The development of fractional order derivatives and new ideas in numerical analysis have made it possible for the researchers to save more details about the development of the behaviour of the dynamical systems through time and to use those details for future traits. Sea lions are the prey and sharks are the predators. The prey population is divided into two categories: immature and mature prey. In this model, suppose that the shark (predator) feeds on both mature and immature sea lions, followed by a Crowley–Martin type functional response (CMFR), and that there would be intra-species competition among sea lions as well as sharks. Caputo fractional derivative is used to highlight the link between mature sea lions, juvenile sea lions, and sharks. In this study, we examine the given solution's positivity, boundedness, existence, and uniqueness and shown permanence of the system is highly dependent on the predation rate, conversion rate, death rate and competition rate. Sharks and sea lions are two key components of the marine environment, specifically decided to focus on these two population groups for research. As shark feeds on both adult and juvenile sea lion, they are coupled with predator–prey interaction to produce a complex combined effect as regulators of their population sizes. Caputo fractional derivative is concluded to glean further knowledge about the dynamics of this interaction. The predator feeds not just on mature prey but also on immature prey because they are unable to defend themselves or hide. On prey-predator dynamics, the effect of fractional derivative as well as the influences of various parameters, including the predation rate of mature sea lions, the rate at which mature sea lions turn into sharks, the rate at which immature sea lions die naturally, the rate at which immature sea lions are preyed upon, and the rate at which mature sea lions compete with one another, have been investigated numerically. The number of sea lions and sharks are directly correlated. One of the most significant findings indicate that the predator population is under control, even while the rate of predation is on the rise since not all predators survive for a variety of reasons, including diseases, competition for food, and its own predators. With the aid of a new fractional operator and projected technique, the analysis of the current dynamical system can assist us in comprehending a variety of models in predator-prey ecosystems that are home to fascinating phenomena**.**

Keywords: Prey-predator model, Mature and immature prey, Predictor-corrector method, Crowley-Martin functional response

Sea lions are marine mammals, characterized by long fore flippers, external ear flaps, short and thick hair, and a big chest and belly. These mammals have ability to walk on all fours. Three species, the Australian sea lion, the Galápagos sea lion and the New Zealand sea lion are listed as endangered. Sea lions are pinnipeds who have three main predators which mainly includes sharks. Many researchers believe that the sea lion can easily swim faster than sharks. However, it is surprise that still these mammals are prey to sharks. The very young (immatured) or those that are sick won't be able to move faster hence they are easily catchable by predators. Sea lions can sense when killer sharks are near by. As every prey has defense mechanism, these mammals also do have such as to get to the edge of the water and onto land. Prey-predator model is used to depict the sea lion-shark dynamics in this work. The predator is the sharks, while the prey are the sea lions. The interaction between predator and prey was initially investigated independently by Lotka and Volterra in order to depict the scenario in real life with relevant assumptions. It has gained a lot more importance and has been widely considered because of its significant part in the ubiquitous presence of mathematical models in ecology, as well as a wide range of applications (Selvam 2020). Ecological modelling aims to produce more accurate mathematical models and better explanations in mathematical language with biological interpretations (Zhang 2022).

The functional response (FR), on the other hand, is a unique and noteworthy component that plays a key role in all predator-prey interactions. When a change in prey density is attached per unit time per predator, it is referred to be a FR (Zhang 2022). Some of the notable functional responses are Holling type I, II, III, and IV FR (Mortoja 2019), Bedington-DeAngelis FR (Beddington 1975, DeAngelis 1975, Jawdat

2023, Liu 2011), Leslie-Gower FR (Wei 2016) and square root FR (Panja 2021). When both prey and predator populations are large, the Crowley Martin functional response (CMFR) is used as a predator interference function. The predator-prey model with CMFR was originally proposed by Crowley and Martin (Martin 1989). The Beddington-Deangelis functional reaction is similar to this functional response. This varies only by one extra term, which describes how predators interfere with one another when prey is densely packed. Meng et al (2014) and Maiti (2019) considered a prey and predator model with CMFR and stage structure for prey, where the authors studied the global stability and persistence of the model. More research related to the CMFR can be found in earlier refrences (Tripathi 2016, Kang et al 2017, Dubey 2018, Ghanbari 2020). Other than the predator-prey dynamics, involvement of CMFR can also be noticed in the works related to Human Immunodeficiency Virus (HIV), viral dynamics models, and other infectious (Kumari 2019, Tripathi et al. 2020, Keeley et al 2021, Pandey et al 2022). Liao (2017) used CMFR to see the delayed dynamics of the phytoplankton-zooplankton system. To perform a non-autonomous stochastic analysis of the prey-predator model (Zhang 2016, Xu 2019), the CMFR is used. Further to analyze bifurcation and chaos control involving two delay times, discrete-time CMFR is used (Liu 2019).

For many scientific societies, studying ecological phenomena has remained a perennially appealing topic. The development of fractional calculus has proven to be a useful tool in this process. Fractional derivatives are trending as one of the most effective and essential tools to the physical systems (Delavari 2012, Prakasha 2019, Veeresha 2019, Gao et al 2020, Baishya 2021a, Achar et al 2022, Abu-Shady 2022, Özköse et al 2022). The objective of this study to develop model that depicts interactions between immature sea lions, mature sea lions, and shark populations, with sea lions acting as prey and shark acting as predator in the frame of the Caputo fractional derivative. In the rest of this paper also discussed positivity, boundedness, conditions of existence and uniqueness, their stability analysis, and several numerical simulations concerning different parameters.

MATERIAL AND METHODS

As Caputo fractional derivatives support the integer-order initial conditions, the fractional derivative in this work is considered in the Caputo sense. Here, $d^\alpha_{t_0}$ denotes the Caputo fractional derivative.

Preliminaries

Definition 1: The fractional order integral operator of Riemann-Liouville is defined as (Podlubny 1998)

$$J^\alpha_v f(v) = \frac{1}{\Gamma(\alpha)} \int_0^v \frac{f(t)}{(v-t)^{\alpha-1}} dt, \alpha > 0,$$
$$J^0 f(v) = f(v).$$

Definition 2: *(Caputo Fractional Derivative)* (Podlubny 1998) *The Caputo derivative with fractional order $0 < \alpha < 1$ of function $g(t)$ is defined by:*
$$D^\alpha_{t_0} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} g'(\tau) d\tau,$$
where $\Gamma(\cdot)$ refers to Gamma function.

Lemma 1: (Li et al. 2017) *Assume that $q(t)$ is a continuous function on $(t_0, +\infty)$ which satisfies*
$$D^\alpha_{t_0} q(t) \le -\lambda q(t) + \xi q(t_0) = q_{t_0},$$
here $0 < \alpha \le 1$, $(\lambda, \xi) \, \varepsilon \, R^2$ and $\lambda \ne 0$ and consider $T_0 \ge 0$ as the initial time. Now,
$$q(t) \le (q(t_0) - \frac{\xi}{\lambda}) E_\alpha[-\lambda(t-t_0)^\alpha] + \frac{\xi}{\lambda}.$$

Mathematical model formulation: In this paper, a three species stage structured sea lion-shark model is proposed by applying the three species Crowley-Martin functional response model in the frame of Caputo fractional derivative. It includes immature sea lions (X_i), mature sea lions (X_m) and shark (Y).

$$D^\alpha_{t_0} X_i = \lambda X_m - (r+\delta) X_i - \zeta_1 Y X_i - \mu X_i^2,$$
$$D^\alpha_{t_0} X_m = r X_i - k X_m^2 - \frac{\gamma_1 X_m Y}{(1+aX_m)(1+bY)} - \delta_1 X_m, \quad (1)$$
$$D^\alpha_{t_0} Y = \frac{\gamma_2 X_m Y}{(1+aX_m)(1+bY)} - \delta_2 Y - \beta Y^2 + \zeta_2 Y X_i.$$

Here, λ is the growth rate of the young population, r is the rate of conversion from immature sea lions to mature sea lions, δ is the natural death rate of immature sea lions, ζ_1 is the rate of interaction between immature sea lions and shark and immature sea lions, μ is intraspecific competition among immature sea lions happens when they have to live in the limited area because of sharks fear. Hence they get limited food which makes to compete even among immature sea lions, k is intraspecific competition among mature sea lions, and γ_1 is the rate of interaction between mature sea lions and sharks. Here, we have assumed that sharks depends on both mature and immature sea lions for food and consumes both. δ_1 is the natural death rate of mature sea lions, and γ_2 is the conversion rate from mature sea lions to shark (as shark survives due to sea lions consumption, shark gives birth to offsprings), δ_2 is the natural death rate of shark, ζ_2 rate of interaction between immature sea lions and shark, β is intraspecific competition among sharks which occurs for food, breeding, etc, a and b are two positive constants that represent the magnitude of interference among sea lions and sharks.

Uniqueness and existence: The existence of the solutions of the proposed model (1) is demonstrated using the fixed-point theorem in this section. Since the model (1) is complex and non-local, there are no accurate

approaches for evaluating the exact solutions. Still, existence can be guaranteed if expected conditions are met. The system (1) can be rewritten as:

$$D_{t_0}^\alpha [X_i(t)] = \mathbb{A}_1(t, X_i),$$
$$D_{t_0}^\alpha [X_m(t)] = \mathbb{A}_2(t, X_m), \tag{2}$$
$$D_{t_0}^\alpha [Y(t)] = \mathbb{A}_3(t, Y).$$

The system (1) can be transformed into Volterra type integral equation as:

$$D_t^\alpha [X_i(t)]X_{i_0}(t) - X_i(t_0) = \frac{1}{\Gamma(\alpha)}\int^t \mathbb{A}_1(\tau, X_{i_0}(\tau))(t-\tau)^{\alpha-1}d\tau,$$

$$X_m(t) - X_m(t_0) = \frac{1}{\Gamma \alpha}\int_{t_0}^t \mathbb{A}_2(\tau, X_m(\tau))(t-\tau)^{\alpha-1}d\tau, \tag{3}$$

$$Y(t) - Y(t_0) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^t \mathbb{A}_3(\tau, Y(\tau))(t-\tau)^{\alpha-1}d\tau.$$

Theorem 1: *In the region $e \times [t_0, T]$, where*

$$\varrho = \quad X_i, X_m, Y \in \mathbb{R} : max \; |X_i|, |X_m|, |Y| \leq \mathfrak{U} ,$$

and $T < +\infty$, the Lipschitz condition is utilized and contraction occurs by the kernel A if $0 \leq (r + \delta + (\zeta_1 + 2\mu)\mathfrak{U}) < 1$ is correct.

Proof: We will take two functions X_i and $\bar{X}_i$ as:
$$||\mathbb{A}_1(t, X_i) - \mathbb{A}_1(t, \bar{X}_i)|| = ||\lambda X_m - (r+\delta)X_i - \zeta_1 YX_i - \mu X_i^2$$

$$- \left(\lambda X_m - (r+\delta)\bar{X}_i - \zeta_1 Y\bar{X}_i - \mu \bar{X}_i^2\right)||$$
$$\leq ||r + \delta + \zeta_1 Y + \mu(X_i + \bar{X}_i)||||X_i(t) - \bar{X}_i(t)|| \tag{4}$$
$$= (r + \delta + (\zeta_1 + 2\mu)\mathfrak{U})||X_i(t) - \bar{X}_i(t)||.$$

Taking $\theta_1 = (r + \delta + (\zeta_1 + 2\mu)\mathfrak{U})$ which implies,

$$||\mathbb{A}_1(t, X_i) - \mathbb{A}_1(t, \bar{X}_i)|| \leq \theta_1||X_i(t) - \bar{X}_i(t)||. \tag{5}$$

Consequently, the Lipschitz condition is satisfied for $\mathbb{A}_1$ and if

$$0 \leq (r + \delta + (\zeta_1 + 2\mu)\mathfrak{U}) < 1, \quad \text{then} \quad \mathbb{A}_1 \quad \text{follows}$$
contraction. Likewise, it can be demonstrated and exemplified in the other circumstances as given below:

$$||\mathbb{A}_2(t, X_m) - \mathbb{A}_2(t, \bar{X}_i)|| \leq \theta_2||X_m(t) - \bar{X}_m(t)||,$$
$$||\mathbb{A}_3(t, Y) - \mathbb{A}_3(t, \bar{Y})|| \leq \theta_3||Y(t) - \bar{Y}(t)||. \tag{6}$$

Where $\theta_2 = \delta_1 + (2k + \gamma_1)\mathfrak{U}$, $\theta_3 = \delta_2 + \zeta_2(2\beta + \gamma_2)\mathfrak{U}$ are the bounded functions if

$$0 < \theta_i < 1, \quad i = 2,3 \quad \text{and then} \quad \mathbb{A}_i, \quad i = 2,3 \quad \text{are the}$$
contraction.

Theorem 2: *The solution of the system (1) exists and will be unique, if we acquire some t_α such that*
$$\frac{1}{\Gamma(\alpha)}\theta_i t_\alpha < 1, \text{ for i=1,2,3.}$$

Proof: This theorem proof is proved in three parts.
1. Using system (3), it is written with recursive form as follows:

$$\mathfrak{M}_{1,n}(t) = X_{i_n(t)} - X_{i_{n-1}(t)}$$
$$= \frac{1}{\Gamma(\alpha)}\int_0^t (\mathbb{A}_1(\tau, X_{i_{n-1}(t)})$$
$$- \mathbb{A}_1(\tau, X_{i_{n-2}(t)}))(x-\tau)^{\alpha-1}d\tau,$$
$$\mathfrak{M}_{2,n}(t) = X_{m_n(t)} - X_{m_{n-1}(t)}$$
$$= \frac{1}{\Gamma(\alpha)}\int_0^t (\mathbb{A}_2(\tau, X_{m_{n-1}(t)})$$
$$- \mathbb{A}_2(\tau, X_{m_{n-2}(t)}))(x-\tau)^{\alpha-1}d\tau,$$
$$\mathfrak{M}_{3,n}(t) = Y_n(t) - Y_{n-1}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (\mathbb{A}_3(\tau, Y_{n-1}) -$$

$$\mathbb{A}_3(\tau, Y_{n-2}))(x-\tau)^{\alpha-1}d\tau. \tag{7}$$

The prerequisities are $X_{i_0}(t) = X_i(0), X_{m_0}(t) = X_m(0), Y_0(t) = Y(0)$. Later applying norm to the equation (7) we get

$$||\mathfrak{M}_{1,n}(t)|| = ||X_{i_n(t)} - X_{i_{n-1}(t)}||$$
$$=$$
$$||\frac{1}{\Gamma(\alpha)}\int_0^t (\mathbb{A}_1(\tau, X_{i_{n-1}}) - \mathbb{A}_1(\tau, X_{i_{n-2}}))(x-\tau)^{\alpha-1}d\tau||,$$
$$\leq$$
$$\frac{1}{\Gamma(\alpha)}\int_0^t ||(\mathbb{A}_1(\tau, X_{i_{n-1}}) - \mathbb{A}_1(\tau, X_{i_{n-2}}))(x-\tau)^{\alpha-1}d\tau||. \tag{8}$$

Using Lipschitz condition, (4), we obtain,
$$||\mathfrak{M}_{1,n}(t)|| \leq \frac{1}{\Gamma(\alpha)}\theta_1\int_0^t ||\mathfrak{M}_{1,n-1}(\tau)d\tau||, \tag{9}$$

Similarly,

$$||\mathfrak{M}_{2,n}(t)|| \leq \frac{1}{\Gamma(\alpha)}\theta_2\int_0^t ||\mathfrak{M}_{2,n-1}(\tau)d\tau||,$$
$$||\mathfrak{M}_{3,n}(t)|| \leq \frac{1}{\Gamma(\alpha)}\theta_3\int_0^t ||\mathfrak{M}_{3,n-1}(\tau)d\tau||. \tag{10}$$

As a result, we can write,

$$X_{i_n(t)} = \sum_{i=1}^n \mathfrak{M}_{1,i}, X_{m_n(t)} = \sum_{i=1}^n \mathfrak{M}_{2,i}, Y_n(t) = \sum_{i=1}^n \mathfrak{M}_{3,i},$$

Applying equations (9) and (10) recursively, we obtain,
$$||\mathfrak{M}_{1,n}(t)|| \leq ||X_{i_n(0)}||[\frac{1}{\Gamma(\alpha)}\theta_1 t]^n,$$
$$||\mathfrak{M}_{2,n}(t)|| \leq ||X_{m_n(0)}||[\frac{1}{\Gamma(\alpha)}\theta_2 t]^n,$$
$$||\mathfrak{M}_{3,n}(t)|| \leq ||Y_n(0)||[\frac{1}{\Gamma(\alpha)}\theta_3 t]^n. \tag{11}$$

As a result, the existence and continuity are proved.
2. To illustarte that the relation equation (11), form the solution for equation (1), we make the following assumptions:
$$X_i(t) - X_i(0) = X_{i_n t} - \mathfrak{T}_{1n}(t),$$
$$X_m(t) - X_m(0) = X_{m_n t} - \mathfrak{T}_{2n}(t), \tag{12}$$
$$Y(t) - Y(0) = Y_n t - \mathfrak{T}_{3n}(t).$$

In order to achieve the expected outcome we set,
$$||\mathfrak{T}_{1n}(t)|| = ||\frac{1}{\Gamma(\alpha)}\int_0^t (\mathbb{A}_1(\tau, x) - \mathbb{A}_1(\tau, x_{n-1}))d\tau||,$$

This gives,
$$||\mathfrak{T}_{1n}|| \leq \frac{1}{\Gamma(\alpha)}\theta_1||x - x_{n-1}||t, \tag{13}$$

Continuing the same procedure recursively, we obtain
$$||\mathfrak{T}_{1n}|| \leq \left(\frac{1}{\Gamma(\alpha)}\theta_1 t\right)^{n+1}\mathfrak{U}, \tag{14}$$

At t_α, we have

$$||\mathfrak{T}_{1n}(t)|| \leq (\frac{1}{\Gamma(\alpha)}\beta_1 t_\alpha)^{n+1}\mathfrak{U}. \tag{15}$$

From equation (15), we can see, as n tends to ∞, $||\mathfrak{T}_{1n}(t)||$ approaches to 0 provided $\frac{1}{\Gamma(\alpha)}$. Similarly, it can be demonstrated that $||\mathfrak{T}_{2n}(t)||, ||\mathfrak{T}_{3n}(t)||$ tends to 0. Hence the proof.
3. Now we will demonstrate the uniqueness for the system of solutions (1). Let us assume that there is a different set of system solutions, say $\hat{X}_i, \hat{X}_m, \hat{Y}$. As a result of the first equation (3),

$$X_i(t) - \hat{X}_i(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (\mathbb{A}_1(t, X_i) - \mathbb{A}_1(t, \hat{X}_i))d\tau,$$

Using the norm, the above equation becomes:

$$||X_i(t) - \hat{X}_i(t)|| = \frac{1}{\Gamma(\alpha)} \int_0^t ||(\mathbb{A}_1(t, X_i) - \mathbb{A}_1(t, \hat{X}_i))d\tau||. \quad (16)$$

By Lipschitz condition,

$$||X_i(t) - \hat{X}_i(t)|| \leq \frac{1}{\Gamma(\alpha)} \theta_1 t ||X_i - \hat{X}_i||,$$

This results is,

$$||X_i(t) - \hat{X}_i(t)||(1 - \frac{1}{\Gamma(\alpha)} \theta_1 t_\alpha) \leq 0.$$

Since $(1 - \frac{1}{\Gamma(\alpha)} \theta_1 t_\alpha) > 0$, we much have $||X_i(t) - \hat{X}_i(t)|| = 0$. This implies $X_i(t) = \hat{X}_i(t)$.

Positivity : In the biological system it is essential to ensure that all the solutions are non negative. Let $\mho_+$ =$X_i, X_m, Y \in \mho: X_i, X_m, Y \in \mathbb{R}_+$ where $\mathbb{R}_+$ is the set of non-negative real numbers. First let us show that $X_i(t) \geq 0$ all t $\in [t_0, T]$. If possible, let us assume that there exist t_2 such that $X_i(t_2) < 0, X_m(t) \geq 0, Y(t) \geq 0$.

Now $X_i(t_0) > 0, X_i(t_2) < 0$ implies that there exists some $t_1 \in (t_0, T]$ such that $X_i(t_1) = 0$, but $X_m(t_1) \geq 0, Y(t_1) \geq 0$. From the system of first equation (1), it implies that $X_i(t_2) = 0$ which contradicts the assumption that $X_i(t_2) < 0$

Therefore, $X_i(t) \geq 0 \forall \in [t_0, T]$. Similarly, we can show that $X_m(t) \geq 0, Y(t) \geq 0 \forall \in [t_0, T]$.

Boundedness : Here determine the solutions of the system (**1**) are bounded.

Theorem: *The solutions of the system* (1) *are uniformly bounded.*

Proof.- Define a function, $\mathfrak{Y}(t) = X_i(t) + X_m(t) + Y(t)$,by using the lemma (1) we get,

$$^C_{t_0}D_t^\alpha \mathfrak{Y}(t) + \delta_1 \mathfrak{Y}(t) = ^C_{t_0} D_t^\alpha [X_i(t) + X_m(t) + Y(t)]$$
$$+ \delta_1 [X_i(t) + X_m(t) + Y(t)]$$

$$= \lambda X_m - (r + \delta)X_i - \zeta_1 YX_i - \mu X_i^2 + rX_i - kX_m^2 -$$
$$\frac{\gamma_1 X_m Y}{(1 + aX_m)(1 + bY)} - \delta_1 Y + \quad (17)$$
$$\frac{\gamma_2 \frac{X}{m} Y}{(1 + aX_m)(1 + bY)} - \delta_2 Y - \beta Y^2 + \zeta_2 YX_i + \delta_1(X_i + X_m + Y)$$

$$\leq \lambda X_m + \zeta_2 YX_i + \gamma_2 X_m Y + \delta_1(X_i + X_m). \quad (18)$$

The solution exists and is unique in
$$\mho = (X_i, X_m, Y)/max\{|X_i|, |X_m|, |Y|\} \leq \mathfrak{P}\},$$
The above inequality yields,
$$^C_{t_0}D_t^\alpha \mathfrak{Y}(t) + \delta_1 \mathfrak{Y}(t) \leq (\lambda + 2\delta_1 + \zeta_2 \mathfrak{P} + \gamma_2 \mathfrak{P})\mathfrak{P},$$
By the lemma 1, we get

$$^C_{t_0}D_t^\alpha \mathfrak{Y}(t) \leq$$

$$(\mathfrak{Y}(t_0) - \frac{1}{\delta_1}((\lambda + 2\delta_1 + \zeta_2 \mathfrak{P} + \gamma_2 \mathfrak{P})\mathfrak{P})E_\alpha[-\eta(t - t_0)^\alpha] +$$

$$\frac{1}{\delta_1}((\lambda + 2\delta_1 + \zeta_2 \mathfrak{P} + \gamma_2 \mathfrak{P})\mathfrak{P}) \rightarrow ((\lambda + 2\delta_1 + \zeta_2 \mathfrak{P} + \gamma_2 \mathfrak{P})\mathfrak{P}),$$

$$t \rightarrow \infty$$

Therefore, all the solution of the system (1) that initiates in $\mho$ remained bounded in
$$\Theta = \{(X_i, X_m, Y) \in \mho_+ | \mathfrak{Y}(t)$$
$$\leq ((\lambda + 2\delta_1 + \zeta_2 \mathfrak{P} + \gamma_2 \mathfrak{P})\mathfrak{P}) + \epsilon, \quad \epsilon$$
$$> 0\}.$$

RESULTS AND DISCUSSION

Here, the points of equilibrium of the system (1) are found.

1. The shark-free equilibrium point $E_1 = (\hat{X}_i, \hat{X}_m, 0)$ exists.

Theorem: *There exists unique positive shark-free equilibrium point* $E_1 = (\hat{X}_i, \hat{X}_m, 0)$

Proof. The component of E_1 can be obtained by solving the following nonlinear system of equations:

$$\lambda \hat{X}_m - (r + \delta)\hat{X}_i - \mu \hat{X}_i^2 = 0, \quad (19)$$

$$r\hat{X}_i - k\hat{X}_m^2 - \delta_1 \hat{X}_m = 0. \quad (20)$$

From equation (20) we get,

$$\hat{X}_i = \frac{kX_m^2 + \delta_1 X_m}{r}. \quad (21)$$

Using equation (21) in equation (19), we get

$$\lambda X_m = \frac{kX_m^2 + \delta_1 X_m}{r}\left(r + \delta + \mu\left(\frac{kX_m^2 + \delta_1 X_m}{r}\right)\right),$$

$$k(X_m + \delta_1) + k\frac{\delta}{r}(X_m + \delta_1) + \mu\frac{k}{r^2}(X_m + \delta_1)(kX_m^2 + \delta_1 X_m) - \lambda = 0,$$

$$X_m^3\left(\frac{\mu k^2}{r^2}\right) + X_m^2(\delta_1 + k\delta_1)\left(\frac{\mu k^2}{r^2}\right) + X_m\left(\delta_1^2\left(\frac{\mu k^2}{r^2}\right) + \frac{l\delta}{r} + k\right)$$
$$-\left(\frac{k\delta\delta_1}{r} + k\delta_1 - \lambda\right) = 0. \quad (22)$$

Equation (22) have a unique positive root. Hence, the system(1) has unique shark-freeequillibrium point.

2. The coexistence equillibrium point $E_2 = (X_i^*, X_m^*, Y^*)$ exists.

Its component exists by solving the following system of equations:

$$\lambda X_m^* - (r + \delta)X_i^* - \zeta_1 Y^* X_i^* - \mu X_i^{2^*} = 0, \quad (23)$$

$$rX_i^* - kX_m^{2^*} - \frac{\gamma_1 X_m^* Y^*}{(1 + aX_m^*)(1 + bY^*)} - \delta_1 X_m^* = 0, \quad (24)$$

$$\frac{\gamma_2 X_m^* Y^*}{(1 + aX_m^*)(1 + bY^*)} - \delta_2 Y^* - \beta Y^{2^*} + \zeta_2 Y^* X_i^* = 0. \quad (25)$$

From equation (23), we get,

$$X_m^* = \frac{r + \delta + \zeta_1 Y^* X_i^* + \mu X_i^{2^*}}{\lambda}, \quad (26)$$

Using the equation (26) into equation (24) and (25) one gets,

$$rX_i^* - k\left(\frac{r + \delta + \zeta_1 Y^* X_i^* + \mu X_i^{2^*}}{\lambda}\right)^2$$
$$-\frac{\gamma_1 Y^*}{(1 + bY^*)}\frac{r + \delta + \zeta_1 Y^* X_i^* + \mu X_i^{2^*}}{(\lambda + a(r + \delta + \zeta_1 Y^* X_i^* + \mu X_i^{2^*}))}$$
$$-\delta_1\left(\frac{r + \delta + \zeta_1 Y^* X_i^* + \mu X_i^{2^*}}{\lambda}\right) = 0, \quad (27)$$

$$\frac{\gamma_2 Y^*}{(1 + bY^*)}\frac{r + \delta + \zeta_1 Y^* X_i^* + \mu X_i^{2^*}}{(\lambda + a(r + \delta + \zeta_1 Y^* X_i^* + \mu X_i^{2^*}))} - \delta_2 Y^* - \beta Y^{2^*} + \zeta_2 Y^* X_i^* = 0. \quad (28)$$

Hence the X_i^* and Y^* values will be calculated from equations (27) and (28) and putting in equation (26), X_m^* can be determined. Traditional procedure of analysing stability of the interior point of equilibrium, leads to complicated steps. Henceforth we analyse it numerically.

Numerical simulation : In this section, the solution profile of the model (1) is studied numerically by using the Adam-Bashforth numerical method (Diethelm 2002).

Average life span of a sea lion and a shark is 20-30 years. We have assumed the natural death rate of sea lions and sharks in the range of 0.03-0.05. Considering the parametr values as $\delta = 0.02, r = 0.15, k = 0.3, \lambda = 20, a = 2.01, \delta_1 = 0.04, \gamma_1 = 1.7, \gamma_2 = 5, \delta_2 = 0.03, \beta = 0.01, b = 1.01, \zeta_1 = 1.8, \mu = 0.001, \zeta_2 = 0.01, X_{m_0} = 0.5, X_{i_0} = 0.5, Y_0 = 0.5.$ For these values, the eigenvalues at E_2 are $(-4.12056, -0.0163328 + i0.336792, -0.0163328 - i0.336792)$. Therefore E_2 (interior equillibrium point) is stable which is observable in Figures. The eigen values of E_1 (shark-free equillibrium point) are $(-4.87059, 2.98032, -0.628312)$ and hence is unstable. From figures, one can observe that the paths or trajectories first oscillate and show an asymptotic behavior which become stable spiral to stable focus. It can be noticed that the fractional-order parameter α plays a significant role. As the fractional order parameter α reduces, we notice that the solution curves of the system (1) converges from stable spiral to stable node. Specifically, when the values of α increases, the solutions of the system show oscillatory dynamics. As immature sea lions decreases, the mature sea lions also decreases since the conversion rate from immature sea lions to mature sea lions decreases. Here, we can see the shark population survival rate by consumption of mature and immature sea lions. In Figure 2 and 3 we have shown the effect of interaction of sea lions and shark(γ_1) for different values of α. It is noticed that increase in the predation rate γ_1 leads to the increase in **the oscillating behavior of the solutions.**
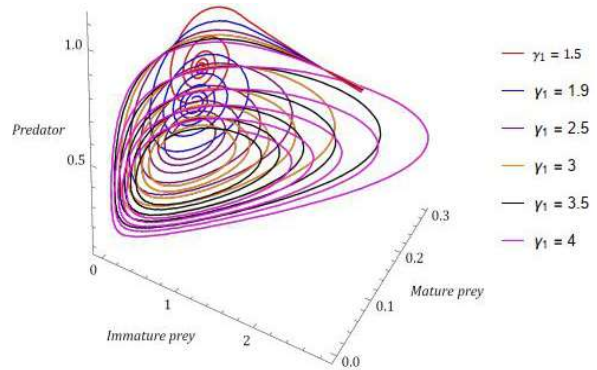

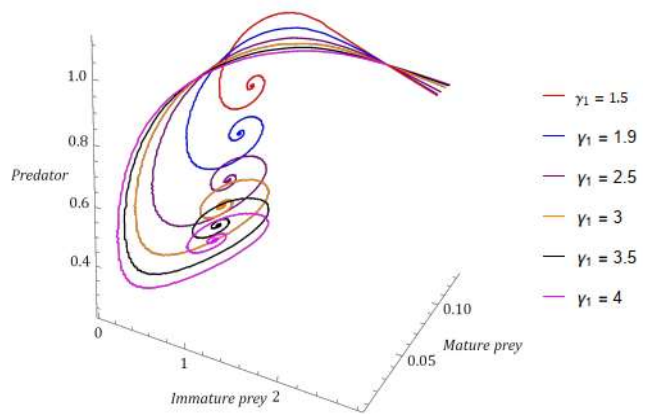**Fig. 2.** Effect of predation rate of mature sea lions by shark for α=1

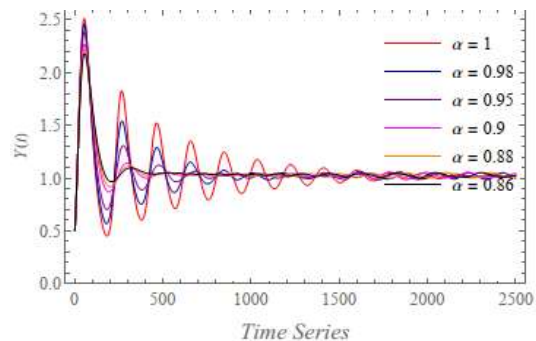**Fig. 3.** Effect of predation rate of mature sea lions by shark for α=0.9
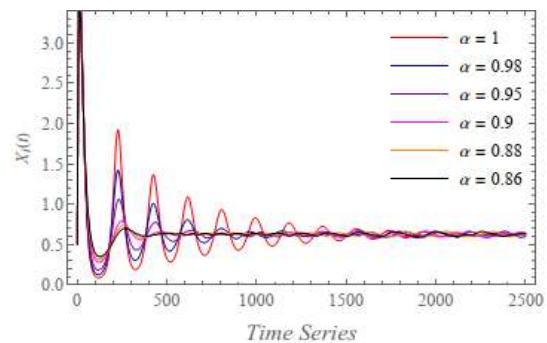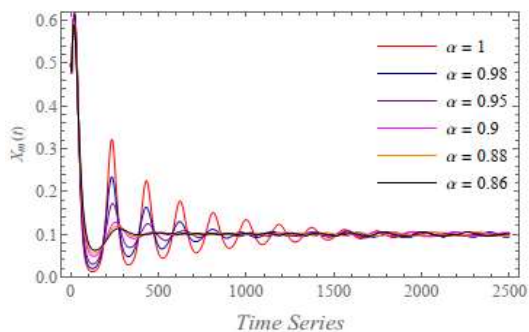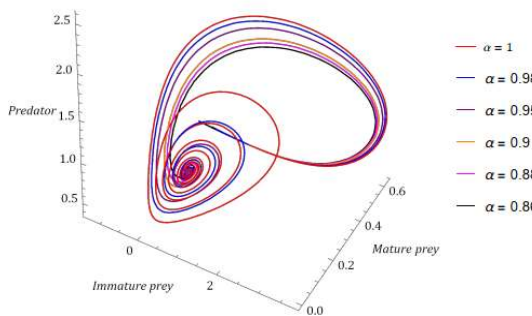

Fig. 1. Numerical simulation for the model with respect to different α

In Figure 4 and 5, we have presented the effect of conversion rate from sea lions to shark (γ_2) for different values of α. It is observed that as the shark population grows extremely, the system observes oscillatory nature.

In Figure 6 and 7, have shown the effect death rate of immature sea lions on all the three populations. Increase in the natural death rate of the immatured sea lions affects the sability of the system. Figure 8 and 9 represent the effect of
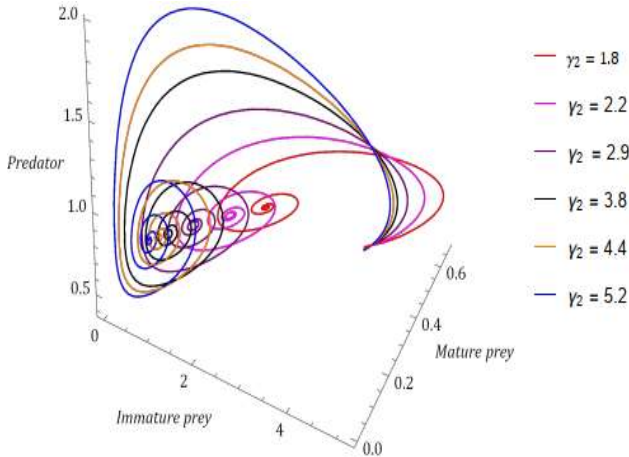

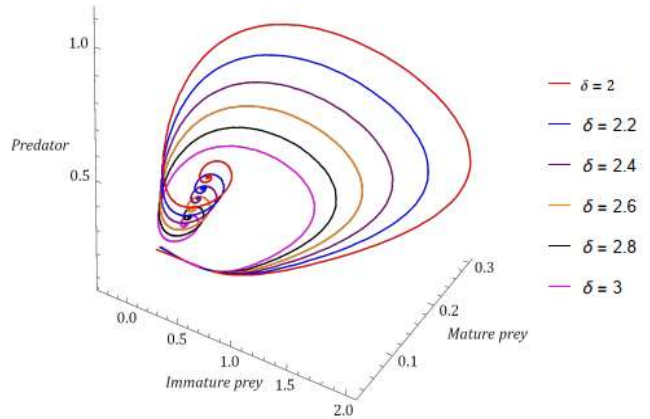**Fig. 4.** Effect of conversion rate from mature from mature sea lions to shark for α=0.98

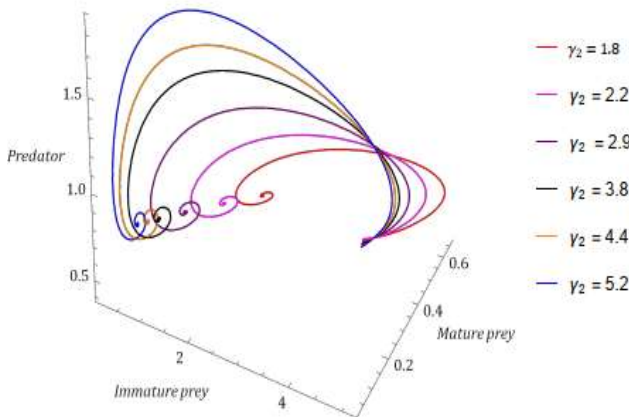**Fig. 7.** Effect of natural death rate of immature sea lions at α=0.9

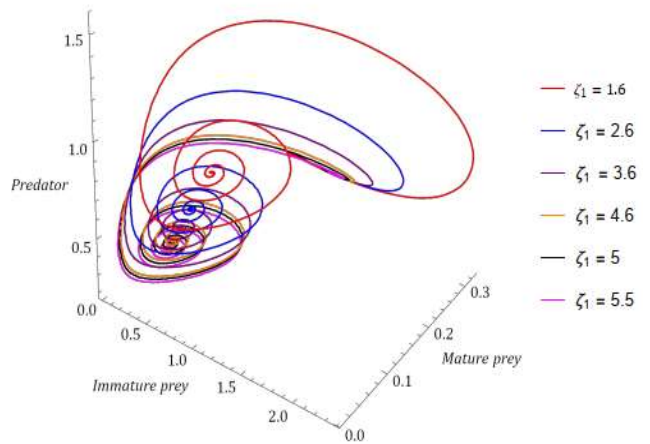**Fig. 5.** Effect of conversion rate sea lions to shark for α=0.9

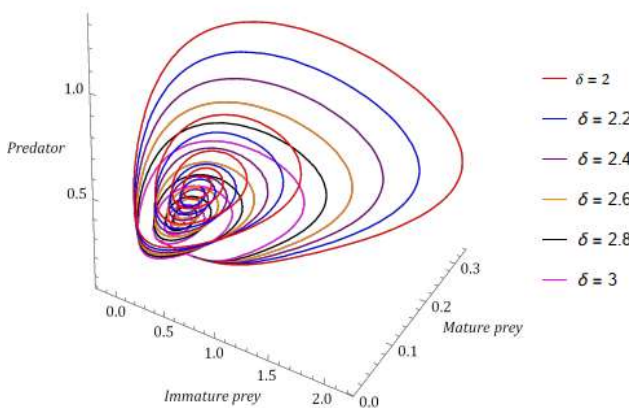**Fig. 8.** Effect of predation rate of sea lions by shark at α=0.98

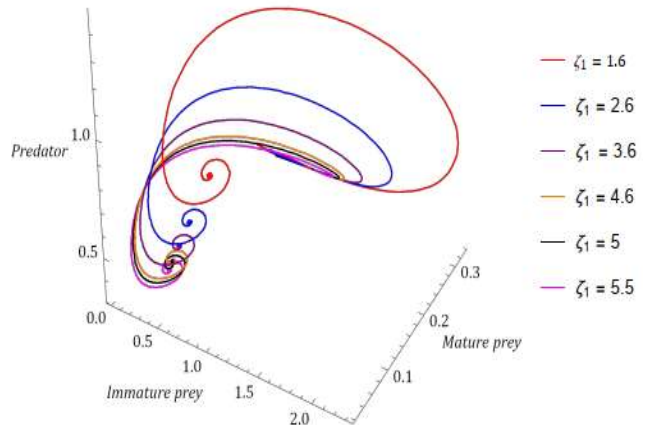**Fig. 6.** Effect of natural death rate of immature sea lions at α=1

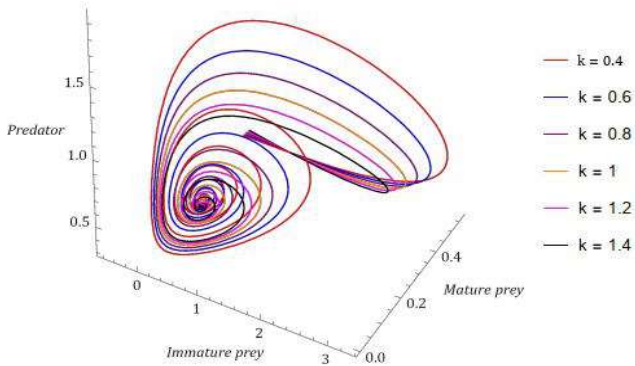**Fig. 9.** Effect of predation rate of immature sea lions by shark at α=0.9

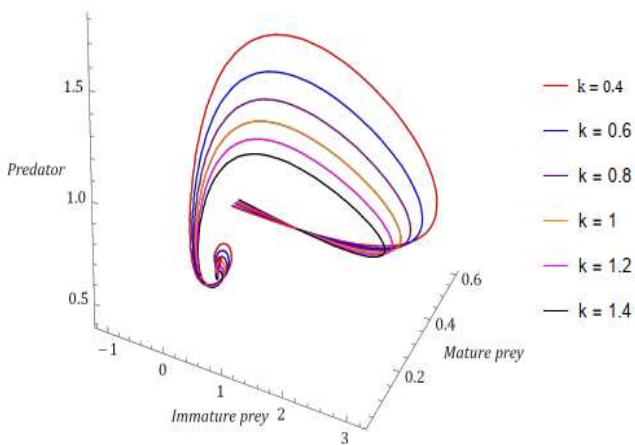**Fig. 10.** Effect of competition rate among mature sea lions at
α=1

Fig. 11. Effect of competition rate among mature sea lions at
α=0.9

interaction among immature sea lions and shark. As immature sea lions have less refuge capacity, sharks attack mostly the immature sea lions. Figure 10 and 11 represent the effect of competition rate among mature sea lions. These figures show that internal competition helps in balancing the populations and fastening the convergence. A common observation made from the numerical simulations is that the presence of fraction derivative influences the stability of the system. It impacts in reducing the oscillatory behaviour of the system.

CONCLUSION

In this paper, a stage structured mathematical model representing the sea lions and shark dynamics has been studied in the framework of Caputo fractional-order derivative. We have used Crowley-Martin functional response to represent the nature of interaction between sea lions and sharks. Important feature of CMFR is that interference effects on feeding rate always remain a prominent factor. A significant point to be noted is that we have introduced competition for food among the immatured

sea lions. Also sharks not only hunts mature sea lions, but also hunt immature sea lions. Key findings from the studies incorporated in this paper are that (a) dynamical behaviour of the population not only depends on the values of the parameters but also on the values of the fractional derivative, (b) the permanence of the system is highly dependent on the predation rate, conversion rate, death rate and competition rate, (c) the numerical simulations reveal that the fractional derivative operator has high influence in stabilizing the solution of the system.

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